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CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL



2014 Trial Examination

# FORM VI

## MATHEMATICS EXTENSION 2

Friday 1st August 2014

### General Instructions

- Reading time — 5 minutes
- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

### Total — 100 Marks

- All questions may be attempted.

### Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

### Section II – 90 Marks

- Questions 11–16 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

### Collection

- Write your candidate number on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eleven.
- Write your candidate number on this question paper and submit it with your answers.

### Checklist

- SGS booklets — 6 per boy
- Multiple choice answer sheet
- Candidature — 79 boys

Examiner

DNW

**SECTION I - Multiple Choice**

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

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**QUESTION ONE**

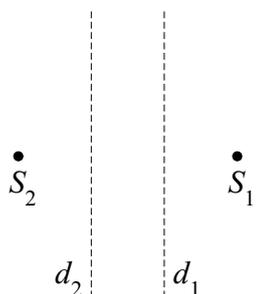
Let  $w = 2 - 3i$  and  $z = 3 + 4i$ . The value of  $w\bar{z}$  is:

1

- (A)  $-6 - 17i$
- (B)  $18 - i$
- (C)  $-6 + 17i$
- (D)  $18 + i$

**QUESTION TWO**

1



A certain conic with eccentricity  $e$  has directrices  $d_1$  and  $d_2$  and foci  $S_1$  and  $S_2$  as shown in the diagram above. The conic is:

- (A) an ellipse because  $e > 1$
- (B) a hyperbola because  $e > 1$
- (C) an ellipse because  $e < 1$
- (D) a hyperbola because  $e < 1$

**QUESTION THREE**

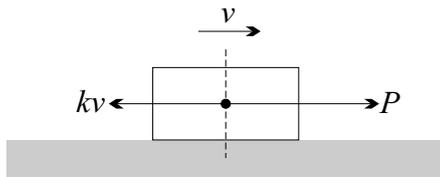
The polynomial  $P(z) = z^3 + (1 + i)z^2 + (1 + i)z + 1$  has a real zero  $z = -1$  and a complex zero  $z = \alpha$ . By considering the sum and the product of the roots, the third root is:

1

- (A)  $\frac{1}{\alpha}$
- (B)  $\bar{\alpha}$
- (C)  $-\alpha$
- (D)  $(1 - \alpha)$

**QUESTION FOUR**

1



The diagram above shows a box of mass  $m$  being pushed with a constant force  $P$  across a floor. The box also experiences a resistive force due to friction which is proportional to its velocity, that is  $kv$ , and is in the opposite direction. There are no other forces acting on the box.

By considering the forces on the box, its limiting velocity is:

- (A) 0
- (B)  $\frac{P}{mk}$
- (C)  $\frac{P}{k}$
- (D)  $\sqrt{\frac{P}{k}}$

**QUESTION FIVE**

1

Which expression is equal to  $\int \frac{1}{\sqrt{8 + 2x - x^2}} dx$ ?

- (A)  $\sin^{-1} \left( \frac{x - 1}{3} \right) + C$
- (B)  $\sin^{-1} \left( \frac{x + 1}{3} \right) + C$
- (C)  $\log \left( (x - 1) + \sqrt{8 + 2x - x^2} \right) + C$
- (D)  $\log \left( (x + 1) + \sqrt{8 + 2x - x^2} \right) + C$

**QUESTION SIX**

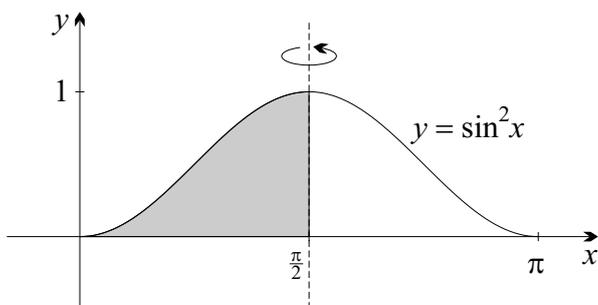
By de Moivre's theorem, the value of  $(1 + i)^{10}$  is:

1

- (A) purely real
- (B) purely imaginary
- (C) a real multiple of  $(1 + i)$
- (D) an imaginary multiple of  $(1 + i)$

**QUESTION SEVEN**

1



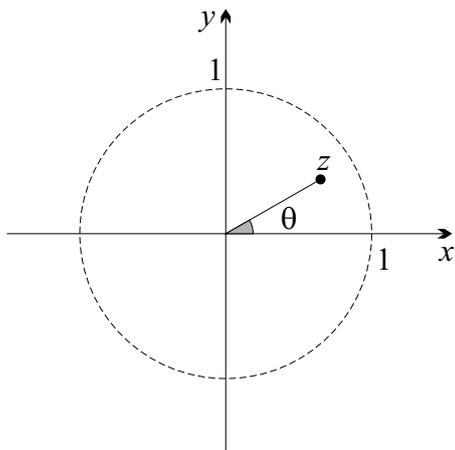
The region below  $y = \sin^2 x$  and above the  $x$ -axis, and between  $x = 0$  and  $x = \frac{\pi}{2}$ , is rotated about the line  $x = \frac{\pi}{2}$  as indicated in the diagram above.

Using the method of cylindrical shells, the integral that represents this volume is:

- (A)  $2\pi \int_0^{\frac{\pi}{2}} (\frac{\pi}{2} - x) \sin^2 x \, dx$
- (B)  $2\pi \int_0^{\frac{\pi}{2}} x \sin^2 x \, dx$
- (C)  $2\pi \int_0^{\frac{\pi}{2}} (\frac{\pi}{2} - x) \sin^2(\frac{\pi}{2} - x) \, dx$
- (D)  $2\pi \int_0^{\frac{\pi}{2}} (\frac{\pi}{2} + x) \sin^2 x \, dx$

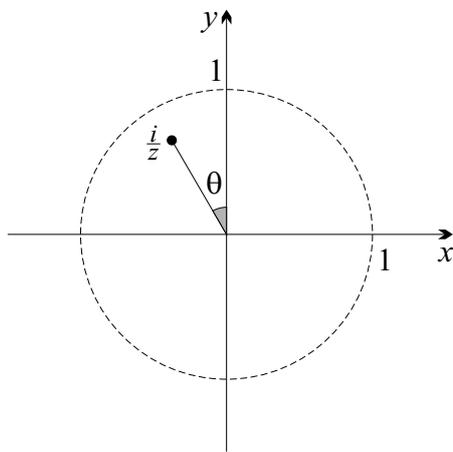
**QUESTION EIGHT**

**1**

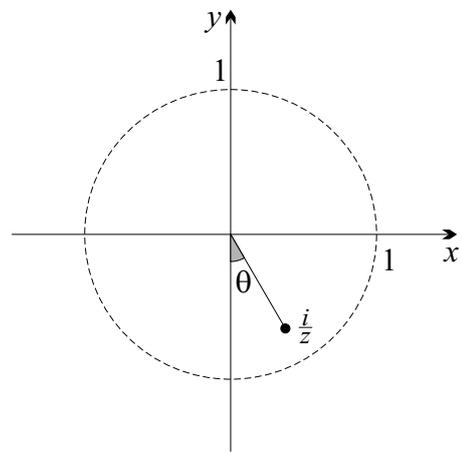


The Argand diagram above shows the complex number  $z$ . By considering the modulus and argument, which diagram below best represents the complex number  $\frac{i}{z}$ ?

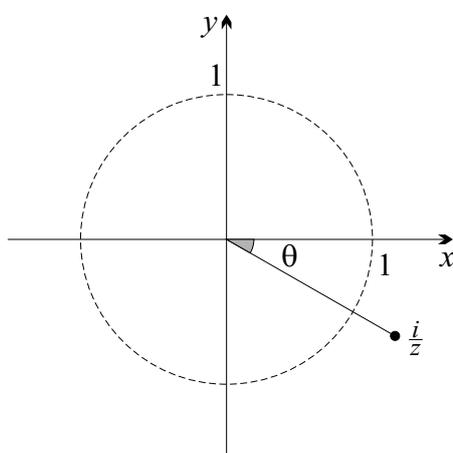
(A)



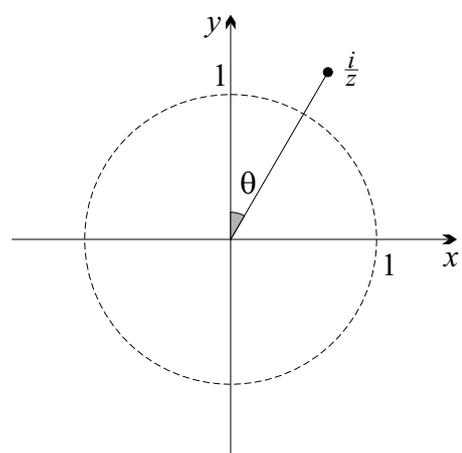
(B)



(C)

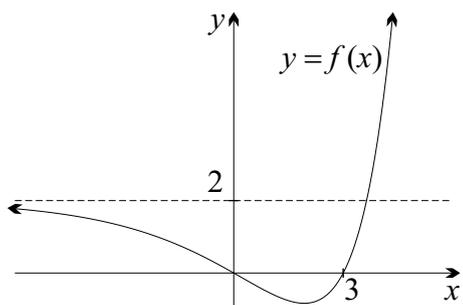


(D)

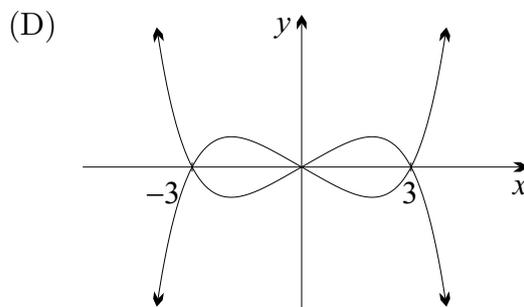
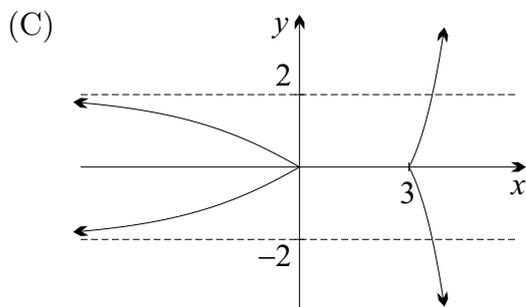
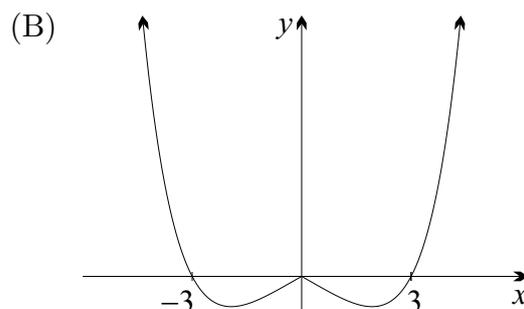
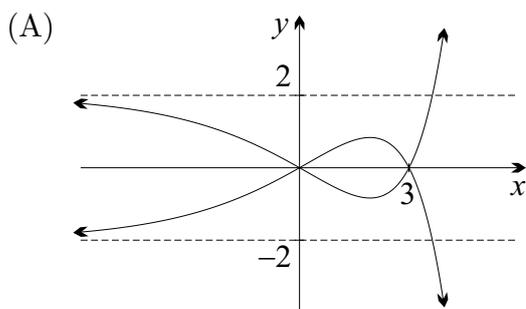


**QUESTION NINE**

1



Above is the graph of  $y = f(x)$ . The correct graph of  $|y| = f(x)$  is:



**QUESTION TEN**

Let  $f(x)$  be an even function where  $f(x) \neq 0$ . Which of the following statements is FALSE? 1

(A)  $\int_{-a}^a f(x) dx = \int_0^a f(x) + f(-x) dx$

(B)  $\int_{-a}^a f(x) dx = \int_0^a f(x) - f(-x) dx$

(C)  $\int_{-a}^a f(x) dx = \int_0^{2a} f(x - a) dx$

(D)  $\int_{-a}^a f(x) dx = \int_0^{2a} f(a - x) dx$

————— End of Section I —————

**SECTION II - Written Response**

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

**QUESTION ELEVEN** (15 marks) Use a separate writing booklet. **Marks**

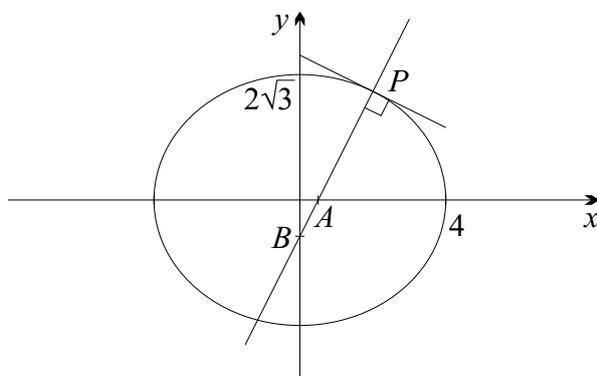
(a) Write  $\frac{2+i}{2-i}$  in the form  $a+ib$ , where  $a$  and  $b$  are real. **2**

(b) Solve the quadratic  $z^2 + 4z + 5 = 0$ . **2**

(c) Find  $\int \frac{2x+1}{2(x+1)} dx$ . **2**

(d) Using the substitution  $t = \tan \frac{\theta}{2}$ , or otherwise, evaluate  $\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin \theta} d\theta$ . **4**

(e)



The point  $P(2, 3)$  lies on the ellipse  $\frac{x^2}{16} + \frac{y^2}{12} = 1$  with eccentricity  $e$ , as shown above.

(i) Show that the equation of the normal at  $P$  is  $2x - y = 1$ . **2**

(ii) The normal at  $P$  intersects the  $x$ -axis at  $A$  and the  $y$ -axis at  $B$ . **3**

Show that  $PA : PB = 1 - e^2 : 1$ .

**QUESTION TWELVE** (15 marks) Use a separate writing booklet. **Marks**

- (a) (i) Find the values of  $A$ ,  $B$  and  $C$  given that **3**

$$\frac{x^2 - 2x + 3}{x^2 - 1} = A + \frac{B}{x - 1} + \frac{C}{x + 1}.$$

- (ii) Hence evaluate **3**

$$\int_2^3 \frac{x^2 - 2x + 3}{x^2 - 1} dx.$$

- (b) Complete the square to find  $\int_{-2}^2 \frac{6}{x^2 + 2x + 4} dx$ . **3**

Give your answer in simplified exact form.

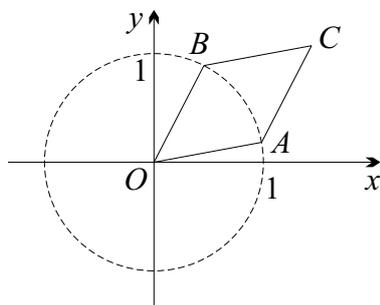
- (c) Evaluate  $15 \int_0^\pi \cos^2 \theta \sin^3 \theta d\theta$ . **3**

- (d) (i) Sketch the region in the  $z$ -plane where  $|z| \leq 2$ . **1**

- (ii) Sketch the region in the  $z$ -plane where  $\frac{3\pi}{4} < \arg(z - 1) \leq \pi$ . **2**

**QUESTION THIRTEEN** (15 marks) Use a separate writing booklet. **Marks**

- (a)



In the Argand diagram above, the points  $A$  and  $B$  lie on the unit circle. The point  $A$  represents the complex number  $z_1 = \text{cis } \alpha$  and  $B$  represents  $z_2 = \text{cis } \beta$ , where  $0 < \alpha < \beta < \frac{\pi}{2}$ . The complex number  $\omega = (z_1 + z_2)$  is represented by the point  $C$ .

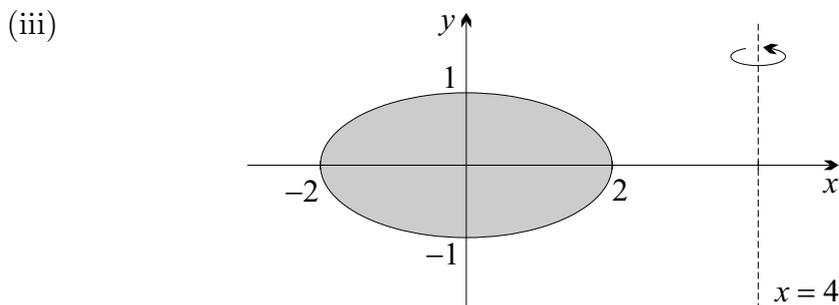
- (i) What type of quadrilateral is  $OACB$ ? **1**  
(You do NOT need to prove your answer.)

- (ii) Let  $\omega = r \text{cis } \theta$ . Use the properties of this quadrilateral to determine  $\theta$  in terms of  $\alpha$  and  $\beta$ . **1**

- (iii) Hence find  $r = |\omega|$  in terms of  $\alpha$  and  $\beta$ . **2**

(b) (i) Without finding a primitive, explain why  $\int_{-a}^a x\sqrt{a^2 - x^2} dx = 0$ . 1

(ii) Without finding a primitive, write down  $\int_{-a}^a \sqrt{a^2 - x^2} dx$ . 1

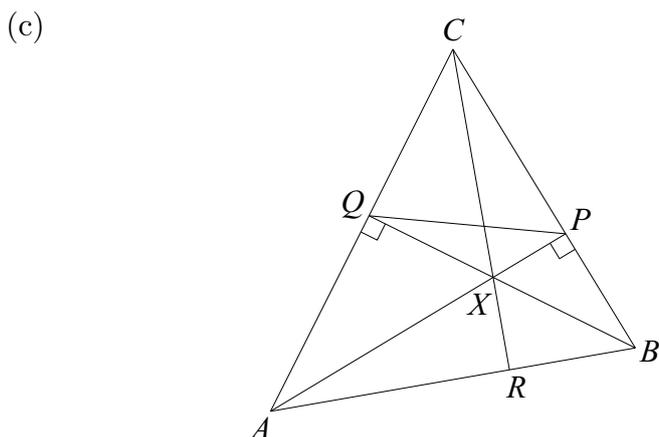


A ring is formed by rotating the ellipse  $\frac{x^2}{4} + y^2 = 1$  about the vertical line  $x = 4$ , as indicated in the diagram above.

(α) Use the method of cylindrical shells to show that the volume  $V$  of this ring is given by the integral 1

$$V = 2\pi \int_{-2}^2 (4 - x)\sqrt{4 - x^2} dx.$$

(β) Hence find  $V$ . 2



The diagram above shows acute angled triangle  $ABC$ . The altitudes  $AP$  and  $BQ$  intersect at  $X$ . The line  $CX$  intersects  $AB$  at  $R$ .

Copy or trace the diagram into your writing booklet.

(i) Give a reason why quadrilateral  $CPXQ$  is cyclic. 1

(ii) Prove that  $\angle CXQ = \angle CPQ$ . 1

(iii) Prove that  $\angle CAB = \angle CPQ$ . 2

(iv) Hence prove that  $CR \perp AB$ . 2

**QUESTION FOURTEEN** (15 marks) Use a separate writing booklet.

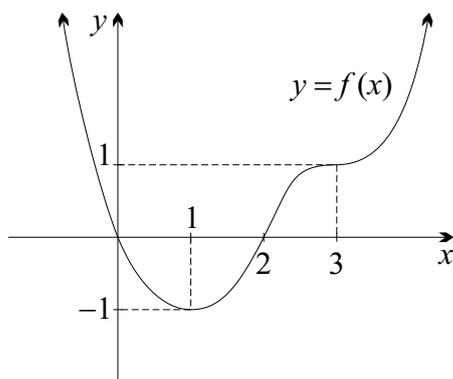
Marks

(a) Let  $f(x)$  be a function with a continuous derivative. Let  $y = \frac{1}{f(x)}$ .

(i) Find  $\frac{dy}{dx}$  in terms of  $f(x)$  and  $f'(x)$ . 1

(ii) Suppose that  $f(a) \neq 0$  and suppose that the graph of  $y = f(x)$  has a horizontal point of inflection at  $x = a$ . Use part (i) to show that the graph of  $y = \frac{1}{f(x)}$  also has a horizontal point of inflection at  $x = a$ . 1

(iii) 3



The diagram above shows the graph of  $y = f(x)$ . Copy or trace the graph into your answer booklet. On the graph in your writing booklet, sketch the graph of  $y = \frac{1}{f(x)}$ , clearly distinguishing it from  $y = f(x)$ .

(b) Let  $z = \text{cis } \theta$ .

(i) Solve  $z^7 = 1$ . 1

(ii) Show that  $\cos n\theta = \frac{1}{2}(z^n + z^{-n})$ . 1

(iii) Suppose that  $\frac{1}{2} + \cos \theta + \cos 2\theta + \cos 3\theta = 0$ . Use part (ii) to show that  $z^7 = 1$ . 2

(iv) Hence solve  $\frac{1}{2} + \cos \theta + \cos 2\theta + \cos 3\theta = 0$  for  $-\pi < \theta \leq \pi$ . 1

(c) The polynomial  $P(z) = z^4 - 4z^3 + 7z^2 - 6z - 4$  has zeroes  $\alpha, \beta, \gamma$  and  $\delta$ .

(i) Show that the polynomial which has zeroes  $(\alpha - 1), (\beta - 1), (\gamma - 1)$  and  $(\delta - 1)$  is 2

$$Q(z) = z^4 + z^2 - 6.$$

(ii) Find the zeroes of  $Q(z)$  and hence find the zeroes of  $P(z)$ . 2

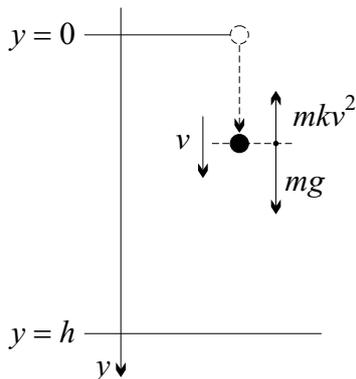
(iii) Hence write  $P(z)$  as the product of linear factors. 1

**QUESTION FIFTEEN** (15 marks) Use a separate writing booklet.

Marks

(a) The polynomial  $P(x) = 4x^3 - 3x - 1$  has a real double zero. Find it. 2

(b)



A mass of  $m$  kilograms falls from rest at a height  $h$  metres above the ground, under the influence of gravity,  $mg \text{ kg m/s}^2$ . As it falls it experiences air resistance equal to  $mkv^2$ , where  $v$  is its speed in metres per second and  $k$  is a positive constant. Let  $y$  metres be the distance it has fallen after  $t$  seconds.

(i) You may assume that as the mass falls 3

$$\ddot{y} = g - kv^2.$$

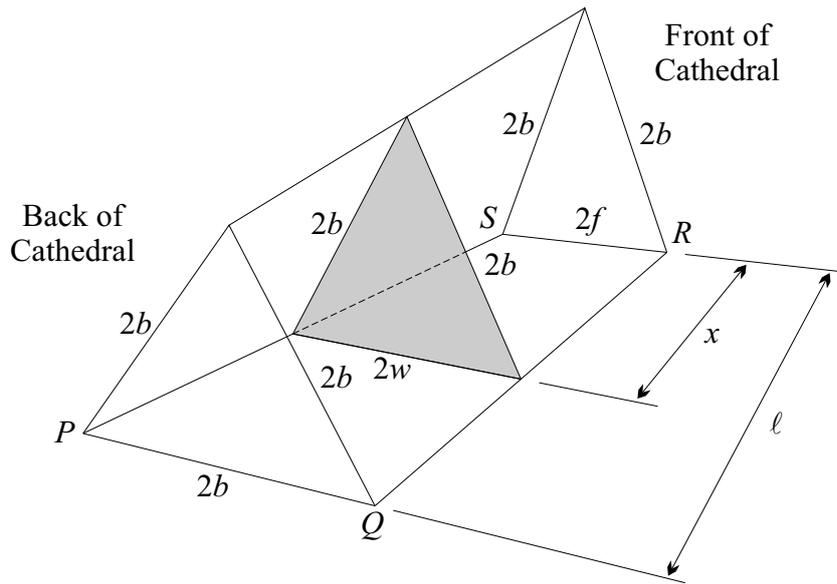
Show that  $v^2 = \frac{g}{k} (1 - e^{-2ky})$ .

(ii) When the mass reaches the ground it has achieved 80% of its terminal velocity. 1

Given that the terminal velocity is  $V_T = \sqrt{\frac{g}{k}}$ , show that  $e^{-2kh} = \frac{9}{25}$ .

(iii) The mass was originally fired upwards from ground level with speed  $u \text{ m/s}$  to reach the height  $h$ . Assuming that the magnitude of the air resistance for the upwards journey is again  $mkv^2$ , show that  $u$  is  $\frac{4}{3}$  of the downwards terminal velocity. 4

(c)



The diagram above shows a simplified model of the roof cavity in the recently built Cardboard Cathedral in Christchurch, New Zealand. At the back of the cathedral the shape of the roof is an equilateral triangle with side length  $2b$ . At the front, the shape is an isosceles triangle with legs  $2b$  and base  $2f$ . The length of the cathedral is  $\ell$ . A typical cross-section, distance  $x$  from the front, is shown shaded. It is an isosceles triangle with legs  $2b$  and base width  $2w$ , and is perpendicular to the base  $PQRS$ . The base is an isosceles trapezium. That is  $PS = QR$  and  $PQ \parallel RS$ .

(i) Use similarity in  $PQRS$  to show that  $w = f + \frac{(b-f)x}{\ell}$ . 2

HINT: Draw a line through  $R$  parallel with  $PS$ .

(ii) Let  $A$  be the area of the typical cross-section, shown shaded. Find  $A$  in terms of  $w$  and  $b$  alone. 1

(iii) The volume of the roof cavity is given by the formula 1

$$V = \int_0^\ell A dx,$$

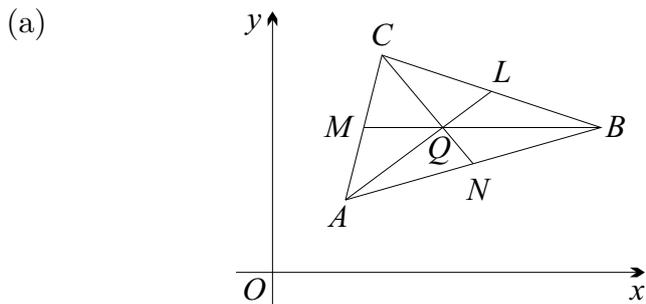
Use the substitution in part (i) and your answer to part (ii) to help show that

$$V = \frac{\ell}{b-f} \int_f^b w \sqrt{4b^2 - w^2} dw.$$

(iv) Hence determine  $V$  in terms of  $b$ ,  $f$  and  $\ell$ . 1

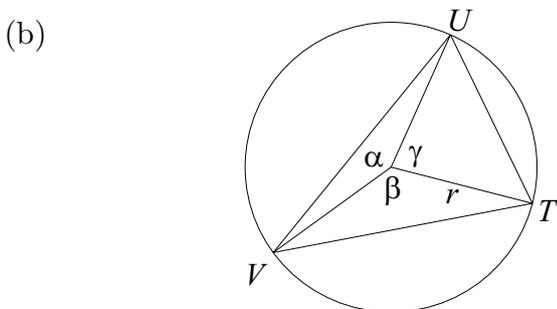
**QUESTION SIXTEEN** (15 marks) Use a separate writing booklet.

Marks



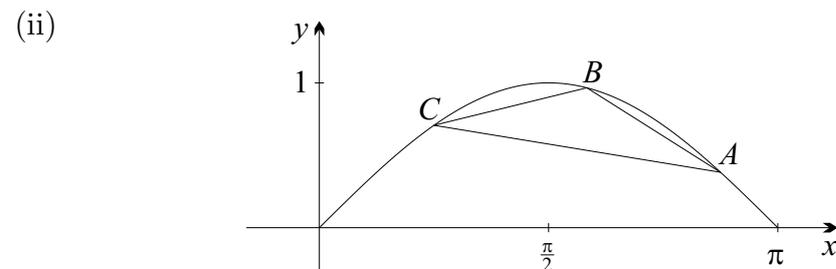
In the Argand diagram above,  $A$  represents the complex number  $z_1$ ,  $B$  represents  $z_2$  and  $C$  represents  $z_3$ . The points  $L$ ,  $M$  and  $N$  are the midpoints of  $BC$ ,  $CA$  and  $AB$  respectively. Intervals  $AL$ ,  $BM$  and  $CN$  are called the medians of  $\triangle ABC$ . They are concurrent at the point  $Q$ , called the centroid. The centroid  $Q$  is always inside  $\triangle ABC$ .

- (i) Write down an expression for the complex number represented by  $L$ . 1
- (ii) Let the point  $W$  represent the complex number  $w = \frac{1}{3}(z_1 + z_2 + z_3)$ .  
By considering  $w - z_1$ , show that  $W$  is on  $AL$ . 2
- (iii) Show that  $Q = W$ . 2



The points  $T$ ,  $U$  and  $V$  lie on a circle of radius  $r$ , as shown in the diagram above. Chords  $UV$ ,  $VT$  and  $TU$  subtend angles  $\alpha$ ,  $\beta$  and  $\gamma$  respectively at the centre of the circle. You may assume that  $0 < \alpha, \beta, \gamma < \pi$ .

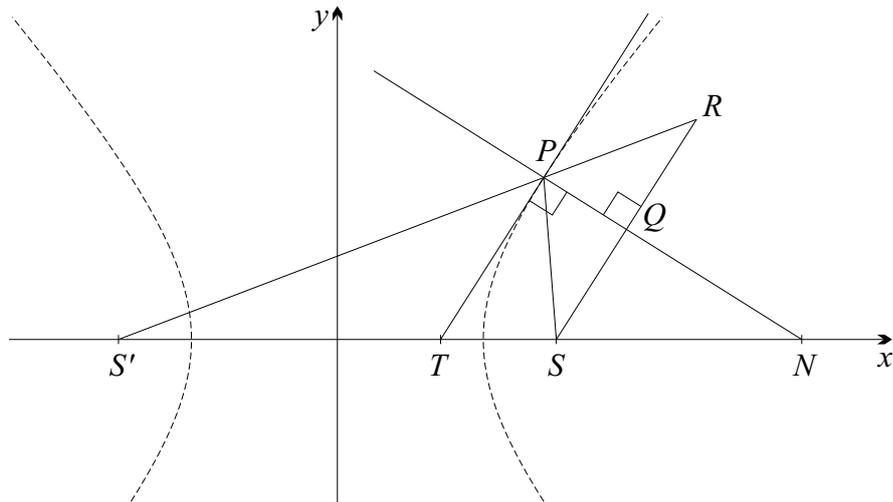
- (i) Write down an expression for the area of  $\triangle TUV$ . 1



The diagram above shows the graph of  $y = \sin x$  for  $0 \leq x \leq \pi$ . Also shown are the points  $A(\alpha, \sin \alpha)$ ,  $B(\beta, \sin \beta)$  and  $C(\gamma, \sin \gamma)$ .

- (a) Use part (a) to write down the coordinates of the centroid of  $\triangle ABC$ . 1
- (b) Hence show that the maximum area of  $\triangle TUV$  occurs when all three angles equal  $\frac{2\pi}{3}$ . 2

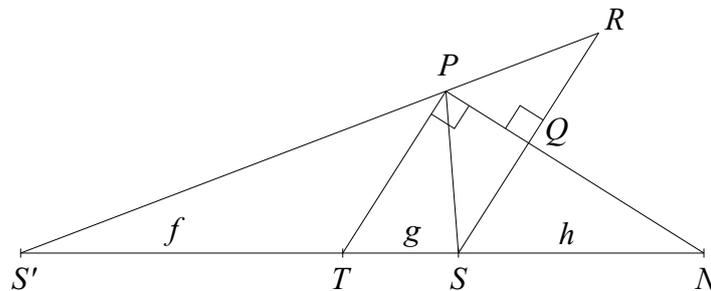
(c)



In the diagram above,  $P(x_1, y_1)$  is a point in the first quadrant on the hyperbola with foci  $S(ae, 0)$  and  $S'(-ae, 0)$ , and eccentricity  $e$ . The tangent at  $P$  intersects the  $x$ -axis at  $T$  and the normal at  $P$  intersects the  $x$ -axis at  $N$ . Let  $PR$  be the reflection of  $PS$  in the line  $PN$ . Let  $SR$  intersect  $PN$  at  $Q$ . Thus  $\triangle PQS \equiv \triangle PQR$ . You may also assume that  $\triangle PTN \parallel \triangle QSN$ .

Do NOT assume that  $S', P$  and  $R$  are collinear.

(i)



2

The triangles have been drawn separately in the diagram above. Let  $S'T$  have length  $f$ ,  $TS$  have length  $g$  and  $SN$  have length  $h$ . By vector addition it should be clear that  $\vec{S'R} = \vec{S'S} + \vec{SR}$ .

Prove that  $\vec{S'R} = \frac{f+g}{f} \vec{S'T} + \frac{2h}{g+h} \vec{TP}$ .

(ii) Given that  $T = (\frac{a^2}{x_1}, 0)$  and  $N = (e^2 x_1, 0)$ , show that

2

$$\frac{2h}{g+h} = \frac{f+g}{f}$$

(iii) Hence prove that  $S', P$  and  $R$  are collinear.

2

End of Section II

**END OF EXAMINATION**

B L A N K P A G E

The following list of standard integrals may be used:

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left( x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left( x + \sqrt{x^2 + a^2} \right)$$

NOTE :  $\ln x = \log_e x, \quad x > 0$



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FORM VI  
MATHEMATICS EXTENSION 2  
Friday 1st August 2014

- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

**Question One**

A  B  C  D

**Question Two**

A  B  C  D

**Question Three**

A  B  C  D

**Question Four**

A  B  C  D

**Question Five**

A  B  C  D

**Question Six**

A  B  C  D

**Question Seven**

A  B  C  D

**Question Eight**

A  B  C  D

**Question Nine**

A  B  C  D

**Question Ten**

A  B  C  D

## Multiple Choice (with comment on errors)

- Q 1 (A) (B)  $wz$  (B)  $\bar{w}z$  (C)  $\overline{(wz)}$
- Q 2 (B) (A)  $e < 1$  for ellipse (C)  $e > 1$  (D)  $e > 1$  for hyperbola
- Q 3 (A) (B) sum not real (C) sum not  $-1$  (D) sum not 0
- Q 4 (C) (A) only if  $P = 0$  (B)  $mkv = P$  (D)  $kv^2 = P$
- Q 5 (A) (B)  $(x + 1)$  wrong (C) not  $\sqrt{a^2 + x^2}$  (D) not  $\sqrt{a^2 + x^2}$
- Q 6 (B) (A)  $10 \times \frac{\pi}{4} \neq n\pi$  (C)  $10 \times \frac{\pi}{4} \neq \frac{\pi}{4} + 2n\pi$  (D)  $10 \times \frac{\pi}{4} \neq \frac{3\pi}{4} + 2n\pi$
- Q 7 (A) (B) about  $x = 0$  (C) incorrect shift (D) about  $x = -\frac{\pi}{2}$
- Q 8 (D) (A)  $iz$  (B)  $-iz$  (C)  $\frac{1}{z}$
- Q 9 (C) (A)  $|y| = |f|$  (C)  $y = f(|x|)$  (D)  $|y| = |f(|x|)|$
- Q 10 (B) (A)  $f$  is even (C) shift right by  $a$  (D) shift right  $a$ , reflect in  $x = a$

## QUESTION ELEVEN (15 marks)

Marks

- (a) Realise the denominator using its conjugate:

$$\begin{aligned} \frac{2+i}{2-i} &= \frac{2+i}{2-i} \times \frac{2+i}{2+i} \\ &= \frac{3+4i}{5} \quad \text{or} \quad \frac{3}{5} + \frac{4}{5}i \end{aligned}$$

 2

- (b) Complete the square (or use another appropriate method.)

$$z^2 + 4z + 5 = 0$$

so  $(z + 2)^2 + 1 = 0$

thus  $(z + 2 + i)(z + 2 - i) = 0$

hence  $z = -2 - i$  or  $-2 + i$

 2

- (c) Separate the fraction to get

$$\begin{aligned} \int \frac{2x+1}{2(x+1)} dx &= \int 1 - \frac{1}{2(x+1)} dx \\ &= x - \frac{1}{2} \log(x+1) + C \quad (\text{or equivalent}) \end{aligned}$$

 2

(d) From  $t = \tan \frac{\theta}{2}$  4

$$d\theta = \frac{2}{1+t^2} dt.$$

at  $\theta = 0, \quad t = 0,$

at  $\theta = \frac{\pi}{3}, \quad t = \frac{1}{\sqrt{3}}$  ✓

so  $\int_0^{\frac{\pi}{3}} \frac{1}{1 - \sin \theta} d\theta = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1 - \frac{2t}{1+t^2}} \times \frac{2}{1+t^2} dt$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{2}{(1-t)^2} dt$$
 ✓

$$= \left[ \frac{2}{1-t} \right]_0^{\frac{1}{\sqrt{3}}}$$
 ✓

$$= \frac{2}{1 - \frac{1}{\sqrt{3}}} - 2$$

$$= 1 + \sqrt{3}$$
 ✓

(e) (i) Differentiate implicitly to get 2

$$\frac{x}{8} + \frac{y}{6} \frac{dy}{dx} = 0$$

so at (2, 3)

$$\frac{dy}{dx} = -\frac{1}{2}$$
 ✓

Thus the normal has gradient 2 and the equation is

$$y - 3 = 2(x - 2)$$
 ✓

or  $y = 2x - 1$

(f) 3

at A  $x = \frac{1}{2}, \quad y = 0$

at B  $x = 0 \quad y = -1$

thus  $PA^2 = \left(\frac{5}{2}\right)^2 + 3^2$   
 $= \frac{45}{4}$

so  $PA = \frac{3}{2}\sqrt{5}.$  ✓

$$PB^2 = 2^2 + 4^2$$

$$= 20$$

so  $PB = 2\sqrt{5}.$  ✓

Hence  $PA : PB = \frac{3}{2}\sqrt{5} : 2\sqrt{5}$

$$= \frac{3}{4} : 1$$

$$= \frac{b^2}{a^2} : 1$$
 ✓

$$= 1 - e^2 : 1$$

**QUESTION TWELVE** (15 marks)

Marks

(a) (i) Rearrange the equation to get

$$x^2 - 2x + 3 = A(x^2 - 1) + B(x + 1) + C(x - 1)$$

at  $x = 1$   $2 = 2B$  so  $B = 1$ .

at  $x = -1$   $6 = -2C$  so  $C = -3$ .

And from the coefficient of  $x^2$ ,

$$A = 1$$

(ii) From part (i)

$$\begin{aligned} \int_2^3 \frac{x^2 - 2x + 3}{x^2 - 1} dx &= \int_2^3 \left( 1 + \frac{1}{x-1} - \frac{3}{x+1} \right) dx \\ &= [x + \log(x-1) - 3\log(x+1)]_2^3 \\ &= 1 + 3\log 3 - 5\log 2 \quad (\text{or equivalent}) \end{aligned}$$

(b) Complete the square in the denominator to get

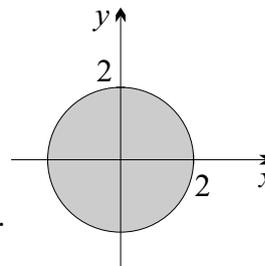
$$\begin{aligned} \int_{-2}^2 \frac{6}{x^2 + 2x + 4} dx &= \int_{-2}^2 \frac{6}{(x+1)^2 + 3} dx \\ &= \frac{6}{\sqrt{3}} \left[ \tan^{-1} \left( \frac{x+1}{\sqrt{3}} \right) \right]_{-2}^2 \\ &= 2\sqrt{3} \left( \frac{\pi}{3} + \frac{\pi}{6} \right) \\ &= \pi\sqrt{3} \end{aligned}$$

(c) By the Pythagorean trigonometric identity

$$\begin{aligned} 15 \int_0^\pi \cos^2 \theta \sin^3 \theta d\theta &= 15 \int_0^\pi \cos^2 \theta (1 - \cos^2 \theta) \sin \theta d\theta \\ &= 15 \int_0^\pi \cos^2 \theta \sin \theta - \cos^4 \theta \sin \theta d\theta \\ &= 15 \left[ \frac{-\cos^3 \theta}{3} + \frac{\cos^5 \theta}{5} \right]_0^\pi \\ &= 15 \left( \left( \frac{1}{3} - \frac{1}{5} \right) - \left( -\frac{1}{3} + \frac{1}{5} \right) \right) \\ &= 4 \end{aligned}$$

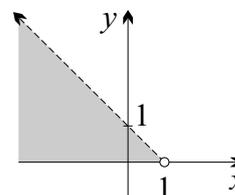
(d) (i)

Inside and on the circle centre the origin, radius 2.



(ii)

The wedge between  $\frac{3\pi}{4}$  and  $\pi$ , with  $\frac{3\pi}{4}$  excluded and with centre 1 excluded.



**Total for Question 12: 15 Marks**

**3**

✓

✓

✓

**3**

✓

✓

✓

**3**

✓

✓

✓

**3**

✓

✓

✓

**1**

✓

**2**

✓

✓

**QUESTION THIRTEEN** (15 marks)

Marks

(a) (i) [Since  $OACB$  is a parallelogram with a pair of equal adjacent sides]  
 $OACB$  is a rhombus.

**1**

(ii)  $OC$  Bisects  $\angle AOB$ ,  
 hence  $\theta = \frac{1}{2}(\alpha + \beta)$

**1**

(iii) Let  $M$  be the mid-point of  $OC$ .

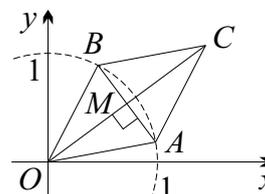
Now  $\angle AOC = \theta - \alpha$   
 $= \frac{1}{2}(\beta - \alpha)$  (from part (ii))

Thus in  $\triangle AOM$

$$OM = \cos\left(\frac{\beta - \alpha}{2}\right) \quad (OA = 1)$$

hence  $r = 2OM$   
 $= 2 \cos\left(\frac{\beta - \alpha}{2}\right)$

**2**



The same result can be obtained by putting  $r \operatorname{cis} \theta = \operatorname{cis} \alpha + \operatorname{cis} \beta$   
 Then divide by  $\operatorname{cis} \theta$  to get  $r = \operatorname{cis}(\alpha - \theta) + \operatorname{cis}(\beta - \theta)$ .  
 Finally, simplify the RHS.

(b) (i) [The region is symmetric and]  
 the integrand is odd.

**1**

(ii) [This integral represents the area of a semi-circle,]  
 hence the value is  $\frac{1}{2}\pi a^2$ .

**1**

(iii) ( $\alpha$ ) The formula is

$$V = 2\pi \int r h dr.$$

In this case  $r = 4 - x$

and from the equation of the ellipse

$$h = 2\sqrt{1 - \frac{x^2}{4}} \quad (\text{twice } y)$$

$$= \sqrt{4 - x^2}$$

$$\text{Thus } V = 2\pi \int_{-2}^2 (4 - x)\sqrt{4 - x^2} dx$$

( $\beta$ )

$$V = 2\pi \int_{-2}^2 (4 - x)\sqrt{4 - x^2} dx$$

$$= 8\pi \int_{-2}^2 \sqrt{4 - x^2} dx - 2\pi \int_{-2}^2 x\sqrt{4 - x^2} dx$$

$$= 8\pi \times \frac{1}{2}\pi 2^2 - 0 \quad (\text{by parts (i) and (ii)})$$

$$= 16\pi^2.$$

**2**

- (c) (i) 1  
 Converse of angles in a semi-circle with diameter  $CX$  ✓
- (ii) 1  
 $\angle CXQ = \angle CPQ$  (angles at the circumference subtended by arc  $CQ$ ) ✓
- (iii) 2  
 $AQPB$  is cyclic (converse of angles in a semi-circle with diameter  $AB$ ) ✓  
 Hence  $\angle CAB = \angle CPQ$  (exterior angle of cyclic quadrilateral  $AQPB$ ) ✓
- (iv) 2  
 Now  $\angle CXQ = \angle CAB$  (by parts (ii) and (iii))  
 hence  $ARXQ$  is cyclic (converse of exterior angle of cyclic quadrilateral) ✓  
 Thus  $\angle ARX = \angle AQX$  (angles in the semi-circle with diameter  $AX$ ) ✓

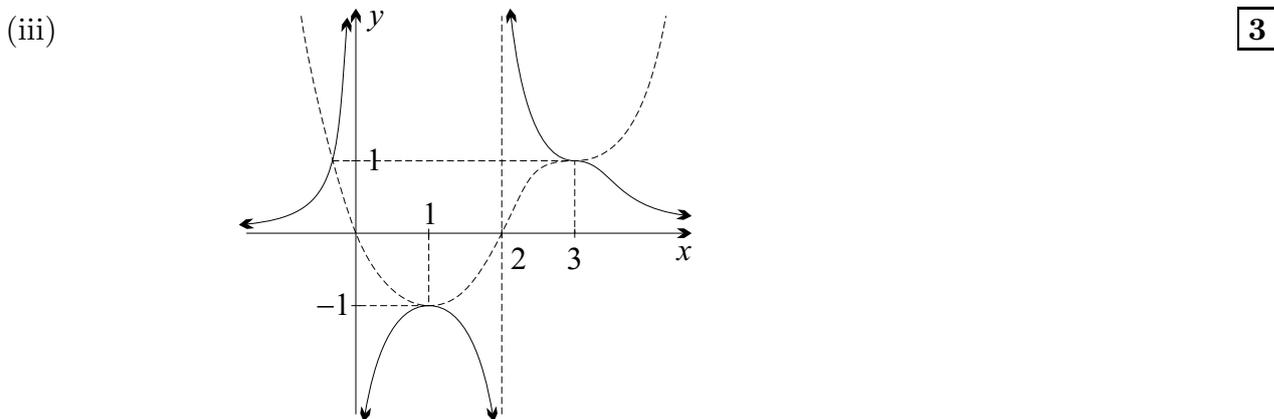
**Total for Question 13:** 15 Marks

**QUESTION FOURTEEN** (15 marks)

Marks

- (a) (i) 1  

$$y = \frac{1}{f(x)}$$
 so 
$$\frac{dy}{dx} = \frac{-f'}{f^2}$$
 ✓
- (ii) 1  
 Since  $f'(a) = 0$ , it follows that  $y' = 0$  at  $x = a$ , and  $y$  has a stationary point.  
 Since  $f$  has an inflexion point at  $x = a$ ,  $f'$  has the same sign each side of  $x = a$ .  
 From part (i),  $y'$  has the opposite sign to  $f'$ , thus  $y'$  also has the same sign each side of  $x = a$ .  
 Hence  $y$  has a horizontal point of inflexion at  $x = a$ . ✓



- maximum at  $(1, -1)$  ✓  
 inflexion at  $(3, 1)$  ✓  
 asymptotes ✓

(b) (i) 1

$$\text{cis } 7\theta = \text{cis } 2n\pi \quad (\text{LHS by de Moivre})$$

so  $\theta = \frac{2n\pi}{7}$  for  $n = 0, \pm 1, \pm 2, \pm 3$  (or equivalent) ✓

(ii) 1

$$\text{RHS} = \frac{1}{2} (\text{cis } n\theta + \text{cis } (-n\theta)) \quad (\text{by de Moivre})$$

$$= \frac{1}{2} (\text{cis } n\theta + \overline{\text{cis } n\theta}) \quad \checkmark$$

$$= \frac{1}{2} \times 2 \cos n\theta$$

$$= \text{LHS}.$$

(iii) Applying part (ii) to the LHS 2

$$\frac{1}{2} (1 + z + z^{-1} + z^2 + z^{-2} + z^3 + z^{-3}) = 0$$

so  $z^{-3} + z^{-2} + z^{-1} + 1 + z + z^2 + z^3 = 0$  ✓

$\times z^3$   $1 + z + z^2 + z^3 + z^4 + z^5 + z^6 = 0$  since  $z \neq 0$ ,

so  $\frac{z^7 - 1}{z - 1} = 0$  (or equivalent) ✓

hence  $z^7 = 1$  where  $z \neq 1$ .

(iv) 1

From parts (i) and (iii) it follows that  $\theta = \frac{2n\pi}{7}$ , for  $n = \pm 1, \pm 2, \pm 3$  ✓

[Penalise inclusion of  $\theta = 0$ , which is clearly NOT a solution.]

(c) (i) 2

$$Q(z) = P(z + 1) \quad \checkmark$$

$$= z^4 + 4z^3 + 6z^2 + 4z + 1 \quad \checkmark$$

$$- 4z^3 - 12z^2 - 12z - 4$$

$$+ 7z^2 + 14z + 7$$

$$- 6z - 6$$

$$- 4$$

---


$$= z^4 + z^2 - 6$$

(ii) 2

$$Q(z) = (z^2 + 3)(z^2 - 2)$$

so the zeroes of  $Q(z)$  are

$$z = \pm\sqrt{2}, \pm i\sqrt{3}. \quad \checkmark$$

Thus the zeroes of  $P(z)$  are

$$z = 1 \pm \sqrt{2}, 1 \pm i\sqrt{3}. \quad \checkmark$$

(iii) 1

Hence  $P(z) = (z - 1 - \sqrt{2})(z - 1 + \sqrt{2})(z - 1 - i\sqrt{3})(z - 1 + i\sqrt{3}).$  ✓

**Total for Question 14:** 15 Marks

**QUESTION FIFTEEN** (15 marks)

Marks

(a) Both  $P(x) = 0$  and  $P'(x) = 0$  at the double zero. 2

Now  $P'(x) = 12x^2 - 3$   
 $= 3(2x - 1)(2x + 1).$

Hence the double zero is either  $\frac{1}{2}$  or  $-\frac{1}{2}$ . √

Since  $P(-\frac{1}{2}) = -\frac{1}{2} + \frac{3}{2} - 1$   
 $= 0,$

the double zero is  $x = -\frac{1}{2}$ . √

(b) (i) 3

$$v \frac{dv}{dy} = g - kv^2$$

or  $\frac{-2kvv'}{g - kv^2} = -2k.$  √

Since the numerator is the derivative of the denominator on the LHS, integrate to get

$$\log(g - kv^2) = -2ky + C \quad (\text{for some constant } C)$$

or  $g - kv^2 = Ae^{-2ky}$  (for some constant  $A$ ) √

Now at  $y = 0, v = 0$  so

$$g = A$$

so  $g - kv^2 = ge^{-2ky}$  √

or  $kv^2 = g(1 - e^{-2ky})$

thus  $v^2 = \frac{g}{k}(1 - e^{-2ky}).$

(ii) At  $y = h, v = \frac{4}{5}\sqrt{\frac{g}{k}}$ , and thus 1

$$\frac{16}{25} \times \frac{g}{k} = \frac{g}{k}(1 - e^{-2kh})$$

thus  $e^{-2kh} = 1 - \frac{16}{25}$   
 $= \frac{9}{25}$  √

(iii) Doing a force balance on the upwards journey, the governing equation is 4

$$\ddot{y} = -g - kv^2$$

so  $\frac{2kvv'}{g + kv^2} = -2k$

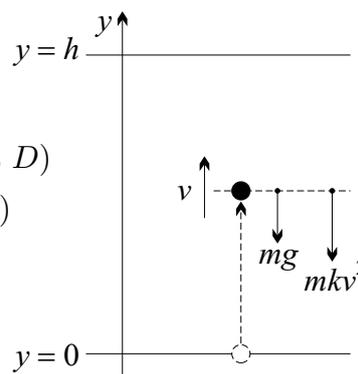
thus  $\log(g + kv^2) = -2ky + D$  (for constant  $D$ ) √

or  $g + kv^2 = Be^{-2ky}$  (for constant  $B$ )

Now at  $y = 0, v = u$  so

$$g + ku^2 = B$$

hence  $g + gv^2 = (g + ku^2)e^{-2ky}$  √



Finally, at  $y = h$ ,  $v = 0$  which gives

$$g = (g + ku^2)e^{-2kh}$$

$$= \frac{9}{25}(g + ku^2) \quad (\text{by part (ii)})$$

so  $ku^2 = g\left(\frac{25}{9} - 1\right)$

and  $u^2 = \frac{16}{9} \times \frac{g}{k}$

hence  $u = \frac{4}{3}\sqrt{\frac{g}{k}}$ .

✓

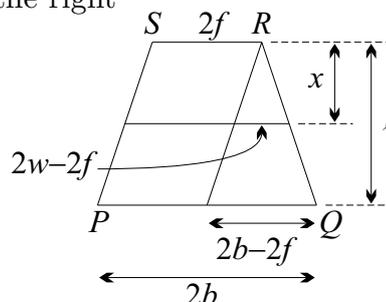
(c) (i) From the similar triangles in the diagram on the right

[Mark for diagram or equivalent working.]

$$\frac{2w - 2f}{2b - 2f} = \frac{x}{\ell}$$

or  $w - f = \frac{(b - f)x}{\ell}$

so  $w = f + \frac{(b - f)x}{\ell}$



2

✓

✓

(ii) The altitude  $h$  of an isosceles triangle bisects the base, so by Pythagoras

$$h^2 = 4b^2 - w^2$$

or  $h = \sqrt{4b^2 - w^2}$

hence  $A = w\sqrt{4b^2 - w^2}$  (half base times height)

1

✓

(iii)

$$dw = \frac{b - f}{\ell} dx \quad (\text{from part (i)})$$

so  $dx = \frac{\ell}{b - f} dw$ .

At  $x = 0$ ,  $w = f$ ,

and at  $x = \ell$ ,  $w = b$ ,

thus  $V = \int_0^\ell A dx$

$$= \frac{\ell}{b - f} \int_f^b w\sqrt{4b^2 - w^2} dw$$

1

✓

(iv)

Hence  $V = \frac{-\ell}{3(b - f)} \left[ (4b^2 - w^2)^{3/2} \right]_f^b$

$$= \frac{\ell}{3(b - f)} \left( (4b^2 - f^2)^{3/2} - (3b^2)^{3/2} \right)$$

[or equivalent.]

1

✓

**Total for Question 15: 15 Marks**

**QUESTION SIXTEEN** (15 marks)

Marks

- (a) (i) From the diagram, since  $L$  is the mid-point of  $BC$  it follows that

**1**

$$L = \frac{1}{2}(z_2 + z_3)$$

✓

- (ii) From the hint,

**2**

$$\begin{aligned} \overrightarrow{AW} &= w - z_1 \\ &= \frac{1}{3}(z_1 + z_2 + z_3) - z_1 \\ &= \frac{1}{3}(z_2 + z_3 - 2z_1). \end{aligned}$$

✓

And 
$$\begin{aligned} \overrightarrow{AL} &= \frac{1}{2}(z_2 + z_3) - z_1 \\ &= \frac{1}{2}(z_2 + z_3 - 2z_1) \\ &= \frac{3}{2}\overrightarrow{AW}. \end{aligned}$$

✓

Since  $\overrightarrow{AW}$  is a real multiple of  $\overrightarrow{AL}$  it follows that  $W$  is on  $AL$ .

- (iii) Similarly,

**2**

$$\begin{aligned} \overrightarrow{BM} &= \frac{1}{2}(z_1 + z_3 - 2z_2) \\ &= \frac{3}{2}(w - z_2) \\ &= \frac{3}{2}\overrightarrow{BW} \end{aligned}$$

✓

Hence  $W$  is on  $BM$ .

Since  $W$  is on both  $BM$  and  $AL$

✓

$$W = Q.$$

- (b) (i) The area of the triangle is

**1**

$$|\Delta UTV| = \frac{1}{2}r^2(\sin \alpha + \sin \beta + \sin \gamma).$$

✓

- (ii) ( $\alpha$ )

**1**

From part (a), the coordinates of the centroid  $Q$  of  $\Delta ABC$  is

$$Q = \left( \frac{1}{3}(\alpha + \beta + \gamma), \frac{1}{3}(\sin \alpha + \sin \beta + \sin \gamma) \right)$$

✓

- ( $\beta$ ) Here is one possible solution.

**2**

Since the angles at the centre add to  $2\pi$ , the coordinates of  $Q$  are

$$Q = \left( \frac{2\pi}{3}, \frac{2}{3r^2}|\Delta TUV| \right)$$

so  $|\Delta TUV|$  is a maximum when  $Q$  is at its highest.

✓

Next,  $y = \sin x$  is concave down for  $0 < x < \pi$ ,

hence  $\Delta ABC$  lies entirely below the curve.

Thus  $Q$  is also below the curve.

The one exception is when  $A = B = C = Q$ .

Finally, in that case  $\alpha = \beta = \gamma$  and  $\alpha + \beta + \gamma = 2\pi$ ,

hence  $Q$  is at its greatest height when  $\alpha = \beta = \gamma = \frac{2\pi}{3}$ .

✓

That is, the greatest area is  $|\Delta TUV| = \frac{3\sqrt{3}r^2}{4}$ .

(c) (i) 2

$$\vec{S'R} = \vec{S'S} + \vec{SR}.$$

By the ratio division of  $S'TS$  it follows that

$$\vec{S'S} = \frac{f+g}{f} \vec{S'T}. \quad \checkmark$$

Also  $\vec{SR} = 2 \vec{SQ}$  (since  $\triangle PQS \equiv \triangle PQR$ ) ✓

$$= \frac{2h}{g+h} \vec{TP} \quad (\text{since } \triangle PTN \parallel \triangle QSN.)$$

Hence  $\vec{S'R} = \frac{f+g}{f} \vec{S'T} + \frac{2h}{g+h} \vec{TP}.$

(ii) From the given coordinates of  $T$  and  $N$ , 2

$$\frac{f+g}{f} = \frac{2ae}{\frac{a^2}{x_1} + ae}$$

$$= \frac{2x_1e}{a + x_1e}. \quad \checkmark$$

Further  $\frac{2h}{g+h} = \frac{2(e^2x_1 - ae)}{e^2x_1 - \frac{a^2}{x_1}}$  ✓

$$= \frac{2x_1e(ex_1 - a)}{e^2x_1^2 - a^2}$$

$$= \frac{2x_1e}{a + x_1e}$$

$$= \frac{f+g}{f}.$$

(iii) Finally, from the previous parts, 2

$$\vec{S'R} = \frac{f+g}{f} \vec{S'T} + \frac{f+g}{f} \vec{TP}$$

$$= \frac{f+g}{f} (\vec{S'T} + \vec{TP})$$

$$= \frac{f+g}{f} \vec{S'P}. \quad \checkmark$$

Since  $\vec{S'R}$  is a multiple of  $\vec{S'P}$ , ✓

it follows that  $S', P$  and  $R$  are collinear.

**Total for Question 16:** 15 Marks

DNW